New Exact Solutions and Localized Excitations in a (2+1)-Dimensional Soliton System

Song-Hua Ma and Jian-Ping Fang

Department of Physics, Zhejiang Lishui University, Lishui 323000, China

Reprint requests to S.-H. M.; E-mail: msh6209@yahoo.com.cn

Z. Naturforsch. **64a,** 37 – 43 (2009); received August 27, 2007 / revised March 3, 2008

Starting from a special conditional similarity reduction method, we obtain the reduction equation of the (2+1)-dimensional dispersive long-water wave system. Based on the reduction equation, some new exact solutions and abundant localized excitations are obtained.

Key words: Conditional Similarity Reduction Method; (2+1)-Dimensional Dispersive Long-Water Wave System; Exact Solutions; Localized Excitations.

PACS numbers: 05.45.Yv, 03.65.Ge

1. Introduction

In the study of nonlinear physics, the search of new exact solutions of nonlinear evolution equations (NEEs) is one of the most important problems. Various methods for obtaining exact solutions to NEEs have been proposed, such as the Lie group method of infinitesimal transformations [1], the nonclassical Lie group method [2], the Clarkson and Kruskal direct method (CK) [3 – 5], the conditional similarity reduction method [6 – 10] and the mapping approach [11 – 16]. In the past, with the help of the improved mapping approach, we have derived some exact excitations of (2+1)-dimensional NEEs, such as (2+1)-dimensional Broer-Kaup-Kupershmidt system, (2+1)-dimensional Boiti-Leon-Pempinelli system, (2+1)-dimensional generalized Broer-Kaup system [17 – 21]. The thought of the mapping approach is based on the reduction theory. Now an important question is whether some simple mapping equations, such as the Riccati equation, can be gotten by the method of conditional similarity reduction, i.e., for a given NEE whether we can transform the NEE to some simple equations which we want to obtain. If yes, new exact excitations of the NEE can be derived based on the exact solutions of these simple equations.

In this paper, we try to extend the conditional similarity reduction method in order to find the conditional similarity reduction equation and some new exact excitations of the (2+1)-dimensional dispersive long-water wave (DLW) system

$$u_{ty}+v_{xx}+u_xu_y+uu_{xy}=0, v_t+(uv)_x+u_{xxy}=0. \quad (1)$$

The DLW system was first derived by Boiti et al. [22] as a compatibility for a "weak" lax pair. In [23], Paquin and Winternitz showed that the symmetry algebra of (1) is infinite-dimensional and has a Kac-Moody-Virasoro structure. Some special similarity solutions are also given in [23] by using symmetry algebra and the classical theoretical analysis. The more general symmetry algebra, W_∞, is given in [24]. In [25], Lou gave out nine types of two-dimensional similarity reductions and thirteen types of ordinary differential equation reductions. In the following, we mainly discuss the conditional similarity reduction and the exact solutions of the (2+1)-dimensional DLW system; then we study the dromion and peakon localized excitations of the system.

2. Conditional Similarity Reduction of the (2+1)-Dimensional DLW System

As is well known, various approaches can be applied to search for the solitary wave solutions of a nonlinear physical model. One of the most efficient methods to find soliton excitations of a physical model is the so-called conditional similarity reduction approach.

First, let us make a restriction for (1):

$$v=u_y. \quad (2)$$

Substituting (2) into (1) yields

$$u_{ty}+u_{xxy}+u_xu_y+uu_{xy}=0. \tag{3}$$

To obtain some special conditional similarity reductions of the (3), we may use the simple ansatz

$$u=f(x,y,t)+g(x,y,t)\phi[q(x,y,t)], \tag{4}$$

where $\phi\equiv\phi(q)$ is the function of the q to be determined, $f\equiv f(x,y,t)$, $g\equiv g(x,y,t)$ and $q\equiv q(x,y,t)$ are functions of (x,y,t) to be determined. Substituting ansatz (4) into (3), we have

$$\begin{aligned}&\sum_i R_iF_i(\phi,\phi',\phi'',\cdots)\\&=R_1\phi'''+R_2\phi\phi''+R_3\phi''+R_4\phi\phi'+R_5\phi'\\&\quad+R_6\phi'^2+R_7\phi^2+R_8\phi+R_9=0,\end{aligned} \tag{5}$$

where $R_i=R_i(x,y,t)$ are ϕ-independent functions, and $F_i=F_i(\phi,\phi',\phi'',\cdots)$ are some polynomials of ϕ and its derivatives, and

$$R_1=gq_yq_x^2, \tag{6}$$

$$R_2=g^2q_yq_x, \tag{7}$$

$$\begin{aligned}R_3=&\,g_yq_x^2+2gq_xq_{xy}+2g_xq_xq_y+gq_yq_t\\&+gq_yq_{xx}+fgq_xq_y,\end{aligned} \tag{8}$$

$$R_4=2gg_xq_y+2gg_yq_x+g^2q_{xy}, \tag{9}$$

$$\begin{aligned}R_5=&\,2g_{xy}q_x+2g_xq_{xy}+g_yq_{xx}+g_tq_y+g_yq_t\\&+g_yq_xf+f_xgq_y+gq_xf_y+g_{xx}q_y+gq_{xxy}\\&+gq_{xy}f+gq_{yt}+g_xq_yf,\end{aligned} \tag{10}$$

$$R_6=R_2=g^2q_yq_x, \tag{11}$$

$$R_7=g_xg_y+gg_{xy}, \tag{12}$$

$$R_8=g_{ty}+f_yg_x+g_{xxy}+f_xg_y+f_{xy}g+fg_{xy}, \tag{13}$$

$$R_9=ff_{xy}+f_{xxy}+f_{yt}+f_xf_y. \tag{14}$$

In the usual CK direct method, function ϕ satisfies only one reduction equation. So, we take some or other coefficient of F_i as the normalizing coefficient of the entire equation. In this paper, we may separate some R into two parts, such as

$$R_3=r_{31}+r_{32},\quad R_5=r_{51}+r_{52}, \tag{15}$$

where

$$r_{31}=g_yq_x^2+2gq_xq_{xy}+2g_xq_xq_y,$$

$$r_{32}=gq_yq_t+gq_yq_{xx}+fgq_xq_y,$$

$$r_{51}=2g_{xy}q_x+2g_xq_{xy},$$

$$\begin{aligned}r_{52}=&\,g_yq_{xx}+g_tq_y+g_yq_t+g_yq_xf+f_xgq_y+gq_xf_y\\&+g_{xx}q_y+gq_{xxy}+gq_{xy}f+gq_{yt}+g_xq_yf,\end{aligned}$$

and separate (5) into three parts. Then we can rewrite (5) as

$$\begin{aligned}&\underbrace{R_1\phi'''+R_2\phi\phi''+r_{32}\phi''+R_6\phi'^2}\\&+\underbrace{R_4\phi\phi'+r_{52}\phi'+r_{31}\phi''}\\&+\underbrace{r_{51}\phi'+R_7\phi^2+R_8\phi+R_9}=0,\end{aligned} \tag{16}$$

and require that the ratios of different derivatives and powers of ϕ in each part are functions of (x,y,t). In other words, taking the R_1 as the normalizing coefficient of the first part the r_{31} as the normalizing coefficient of the second part and the r_{51} as the normalizing coefficient of the third part, we have

$$R_6=R_2=\Gamma_1R_1, \tag{17}$$

$$r_{32}=\Gamma_2R_1, \tag{18}$$

$$R_4=\Gamma_3r_{31}, \tag{19}$$

$$r_{52}=\Gamma_4r_{31}, \tag{20}$$

$$R_7=\Gamma_5r_{51}, \tag{21}$$

$$R_8=\Gamma_6r_{51}, \tag{22}$$

$$R_9=\Gamma_7r_{51}, \tag{23}$$

where Γ_i $(i=1,2,\cdots)$ are some arbitrary functions of q to be determined. In the determinations of f, g, ϕ, and q, as in the usual CK direct method [3,4], we can use some rules to simplify the calculations.

Rule 1. If $f(x,y,t)$ has the form $f=f_0(x,y,t)+g(x,y,t)\Omega(q)$, we can take $\Omega\equiv0$.

Rule 2. If $g(x,y,t)$ has the form $g=g_0(x,y,t)\Omega(q)$, we can take $\Omega\equiv C$ = constant.

Rule 3. If $q(x,y,t)$ is determined by an equation of the form $\Omega(q)=q_0(x,y,t)$, where Ω is an invertible function, we can take $\Omega(q)=q$.

Applying Rule 2 to (17) and supposing that $q_yq_x\neq0$, $\Gamma_1=-2a_2$, we have

$$g=-2a_2q_x, \tag{24}$$

where a_2 is an arbitrary constant. Substituting (24) into (18) and applying Rule 2, we have

$$\Gamma_2 = -a_1, \quad f = -\frac{a_1 q_x^2 + q_t + q_{xx}}{q_x}, \tag{25}$$

where a_1 is an arbitrary constant. Substituting (24) and (25) into (19) – (22), we have

$$\Gamma_3 = 2\Gamma_5 = -2a_2, \quad \Gamma_4 = \Gamma_6 = -a_1. \tag{26}$$

Substituting (24) and (25) into (23) and applying Rule 2, we take $\Gamma_7 = -a_0$ (a_0 is an arbitrary constant). After some detailed calculations, (23) becomes

$$\begin{aligned} &q_x^3(q_{ytt} - q_{yt}) + q_x^3(q_{xx} + q_t)_{xxy} \\ &+ 2q_{xy}q_x(q_{xt} + q_{xxx})(q_t + 2q_{xx}) \\ &- q_x^2(q_{xxx}q_t)_y + q_x q_t q_{xx}(q_t + 2q_{xx})_y \\ &- 2q_x^2(q_{xx}q_{xt} + q_{xx}q_{xxx})_y \\ &+ q_x^4(q_x q_{xx})_y(4a_0a_2 - a_1^2) \\ &+ 9q_{xx}^2(q_x q_{xxy} - q_{xy}) + q_x^2 q_{xy}(q_{xxt} - q_t t) \\ &- q_x^2(q_{xt}q_t)_y + 2q_{xx}^2(q_x q_{yt} - 3q_{xy}q_t) \\ &- q_x^2 q_{xxxx}q_{xy} + q_t^2(q_x q_{xxy} - 3q_{xx}q_{xu}) = 0. \end{aligned} \tag{27}$$

Substituting (24) – (26) and the solutions of (27) into (16), we obtain the similarity reduction equation of (1), which reads

$$\begin{aligned} &-\frac{1}{2}a_2\left[q_x^3 q_y \partial_{qq} + \left(3q_{xy}q_x^2 + q_x q_y q_{xx}\right)\partial_q \right. \\ &\qquad \left. + \left(q_{xx}q_{xy} + q_x q_{xxy}\right)\right] \\ &\cdot \left(\phi' - a_0 - a_1\phi - a_2\phi^2\right) = 0. \end{aligned} \tag{28}$$

Obviously, when the factor a_2 in (28) is not zero, (28) becomes the Riccati equation

$$\phi' = a_0 + a_1\phi + a_2\phi^2. \tag{29}$$

3. New Exact Solutions of the (2+1)-Dimensional DLW System

Of course, it is very difficult to obtain the general solution of (27). Fortunately, in this special case, one of special solutions can be expressed

$$q = \chi(x,t) + \varphi(y), \tag{30}$$

where $\chi \equiv \chi(x,t)$, $\varphi \equiv \varphi(y)$ are two arbitrary variable separation functions of (x,t) and y. Based on (2), (4), (24), (25), (30) and the following type solutions [15] of (29):

(a) periodic solution:

$$\phi = A + B\tan C(q - D), \tag{31}$$

(b) nonperiodic regular solution:

$$\phi = A + B\tanh C(q - D), \tag{32}$$

(c) nonperiodic singular solution:

$$\phi = A + B\coth C(q - D), \tag{33}$$

(d) rational function solution:

$$\phi = Aq + B, \tag{34}$$

where A, B, C, D are arbitrary constants, we obtain the following new exact solutions to the (2+1)-dimensional DLW system:

$$u_1 = -\frac{\chi_t + \chi_{xx}}{\chi_x} + \frac{2\chi_x(1 + \tan(\chi + \varphi))}{\tan(\chi + \varphi) - 1}, \tag{35}$$

$$v_1 = -\frac{4\chi_x\varphi_y \sec(\chi + \varphi)^2}{(1 - \tan(\chi + \varphi))^2}, \tag{36}$$

$$u_2 = -\frac{\chi_t + \chi_{xx}}{\chi_x} + \frac{8\chi_x \tanh(\chi + \varphi)}{1 + \tanh(\chi + \varphi)^2}, \tag{37}$$

$$v_2 = \frac{8\chi_x\varphi_y \operatorname{sech}(\chi + \varphi)^4}{(1 + \tanh(\chi + \varphi)^2)^2}, \tag{38}$$

$$u_3 = \frac{2\chi_x^2 \coth(\chi + \varphi) - \chi_t - \chi_{xx}}{\chi_x}, \tag{39}$$

$$v_3 = -2\chi_x\varphi_y \operatorname{csch}(\chi + \varphi)^2, \tag{40}$$

$$u_4 = -\frac{\chi_t + \chi_{xx}}{\chi_x} + \frac{2a_2\chi_x}{a_2\chi + a_2\varphi + c_0}, \tag{41}$$

$$v_4 = -2\frac{a_2^2\chi_x\varphi_y}{(a_2\chi + a_2\varphi + c_0)^2} \tag{42}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$, where c_0 is an arbitrary constant.

4. Dromion and Peakon Localized Excitations in the DLW System

Due to the arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ included in the above cases, the physical quantities u and v may possess rich structures. For example, when $\chi = ax + ct$ and $\varphi = ky$, all the solutions of

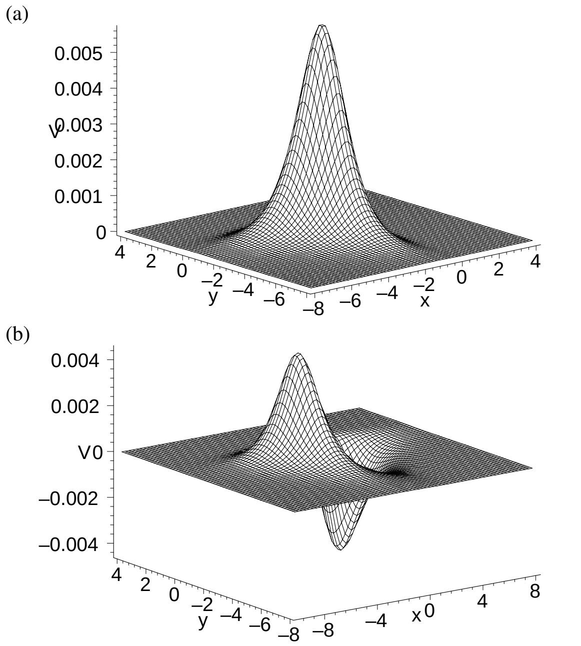

Fig. 1. (a) A plot of a single dromion structure for the physical quantity V with the choice (44) and $c = 1$, $t = 0.5$. (b) A structure of multi-dromions for the physical quantity V with the choice (45) and $c = 1$, $t = 0.2$.

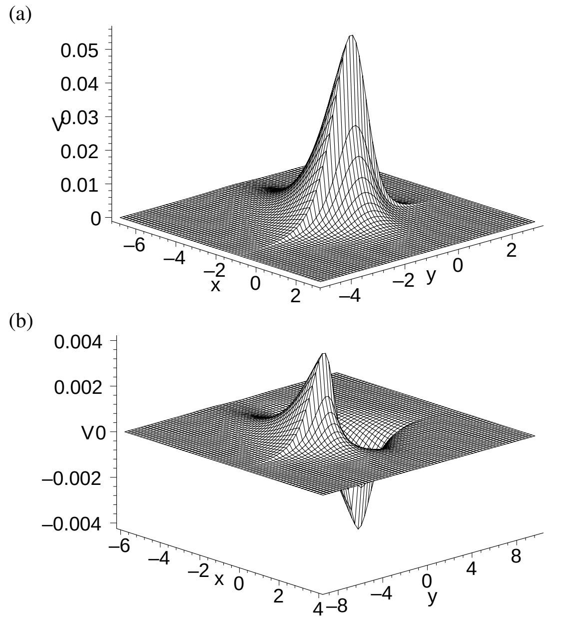

Fig. 2. (a) A plot of a single peakon structure for the physical quantity V with the choice (46) and $c = 1$, $t = 0.8$. (b) A structure of multi-peakons for the physical quantity V with the choice (47) and $c = 1$, $t = 0.5$.

the above cases become simple travelling wave excitations. Moreover, based on the derived solutions, we may obtain rich localized structures such as dromions and peakons. In the following discussion, we merely analyze some special localized excitations of solution v_3 (40), namely

$$V = v_1 = -2\chi_x \varphi_y \operatorname{csch}(\chi + \varphi)^2. \tag{43}$$

4.1. Dromion Localized Excitations

In (2+1)-dimensions, one of the most important nonlinear solutions is the dromion excitation, which is localized in all directions exponentially. For instance, if we choose χ and φ as

$$\chi = 1 + \exp(x + ct), \quad \varphi = 1 + \exp(y), \tag{44}$$

we obtain a simple dromion structure for the physical quantity V (43) presented in Fig. 1a with the fixed parameters $c = 1$ and $t = 0.5$. If we choose χ and φ as

$$\chi = 1 + \operatorname{sech}(x + ct), \quad \varphi = 1 + \exp(y), \tag{45}$$

then we obtain a structure of multi-dromions for the physical quantity V presented in Fig. 1b with the fixed parameters $c = 1$ and $t = 0.2$.

4.2. Peakon Localized Excitations

According to the solution V (43), when the functions χ and φ are selected to be

$$\chi = 1 + \tanh(-|x + ct|), \quad \varphi = 1 + \tanh(y), \tag{46}$$

we obtain a single peakon structure for the physical quantity V presented in Fig. 2a with the fixed parameters $c = 1$ and $t = 0.8$. If we choose χ and φ as

$$\chi = 1 + \operatorname{sech}(-|x + ct|), \quad \varphi = 1 + \tanh(y), \tag{47}$$

we obtain a structure of multi-peakons for the physical quantity V presented in Fig. 2b with the fixed parameters $c = 1$ and $t = 0.5$.

Furthermore, if we choose χ and φ as

$$\begin{aligned} \chi &= 1 + 2\exp(-|x + ct + 2|) \\ &\quad + 1.6\exp(-|x + ct - 1|), \\ \varphi &= 1 + \exp(-|y - 1|), \end{aligned} \tag{48}$$

and

$$\begin{aligned} \chi = 1 &+ 0.5\exp(-|x + ct + 1|) + 0.9\exp(-|x + ct - 1|) \\ &+ 0.4\exp(-|x + ct - 3|) + 0.8\exp(-|x + ct - 5|), \end{aligned}$$

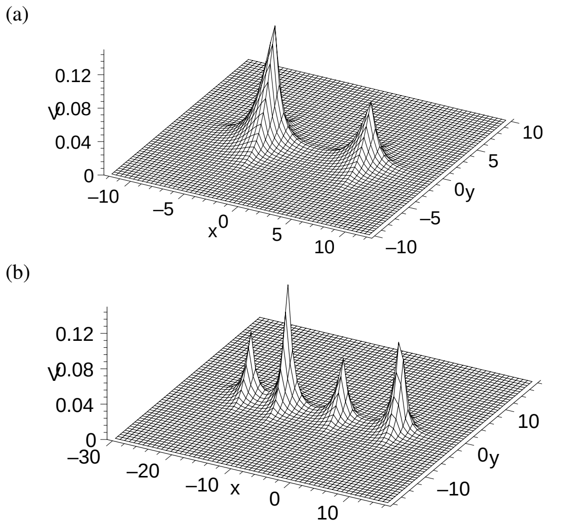

Fig. 3. A structure of another type of multi-peakons for the physical quantity V with (a) the choice (48) and (b) the choice (49) and $c = -2$, $t = 1$.

$$\varphi = 1 + \exp(-|y-1|), \tag{49}$$

we obtain another type of multi-peakons excitation for the physical quantity V presented in Fig. 3a and Fig. 3b with the fixed parameters $c = -2$ and $t = 1$.

4.3. Interactions between Two Solitons

Generally, the interactions between solitons may be regarded as completely elastic. For instance, when $\chi(x,t)$ and $\varphi(y)$ are considered to be

$$\begin{aligned} \chi &= 1 + 0.2\text{sech}(x+ct) + 0.4\text{sech}(x-ct), \\ \varphi &= 1 + \text{sech}(y), \end{aligned} \tag{50}$$

and $c = 1$ in (43), we obtain the interactions between two dromions. Figure 4 shows an evolutional profile corresponding to the physical quantity V expressed by (43). From Fig. 4 and through detailed analysis, we find that the shapes, amplitudes and velocities of the two dromions are completely conserved after their interactions.

For some specific cases the interactions between solitons are nonelastic. For example, if χ and φ are chosen to be

$$\begin{aligned} \chi &= 1 + 1.5\text{csch}(-|x+ct-1|) \\ &\quad + 0.4\text{csch}(-|x+0.3ct-1|), \\ \varphi &= 1 + \tanh(-|y-1|), \end{aligned} \tag{51}$$

and $c = 1$ in (43), we obtain another type of solitary wave solution of (1) with nonelastic behaviour. The

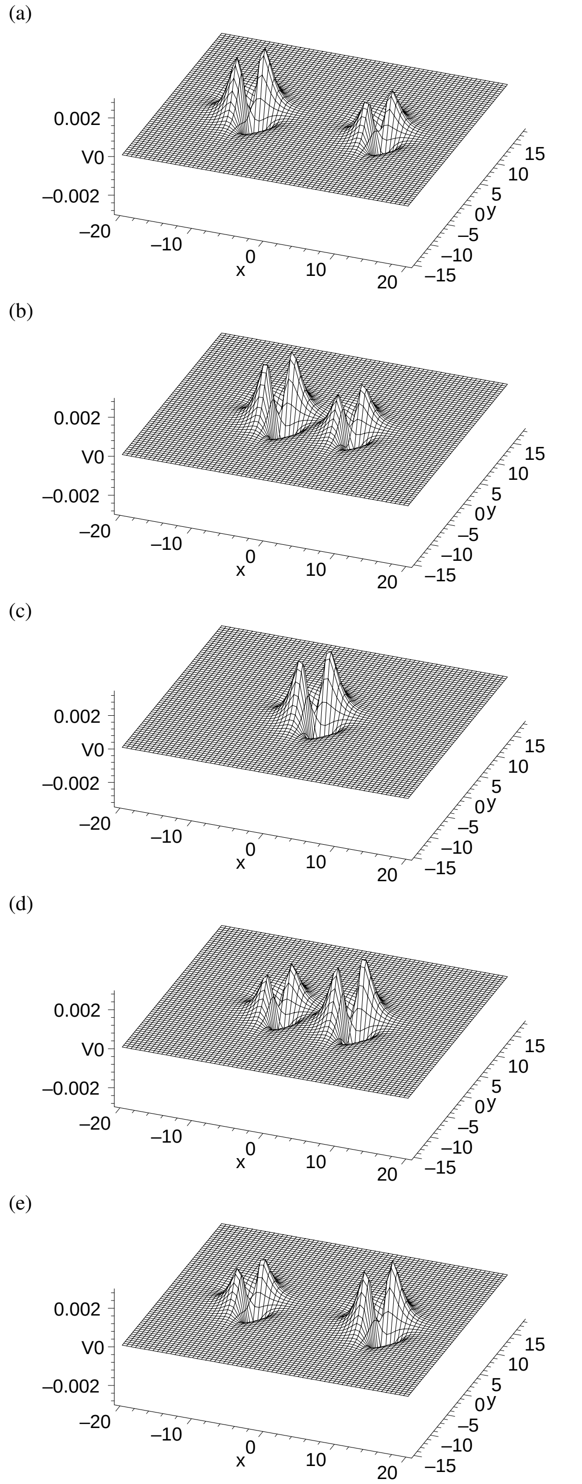

Fig. 4. The evolutional profile of two dromions for the solution V with the condition (50) at different times: (a) $t = -9$, (b) $t = -4$, (c) $t = 0$, (d) $t = 4$, (e) $t = 9$.

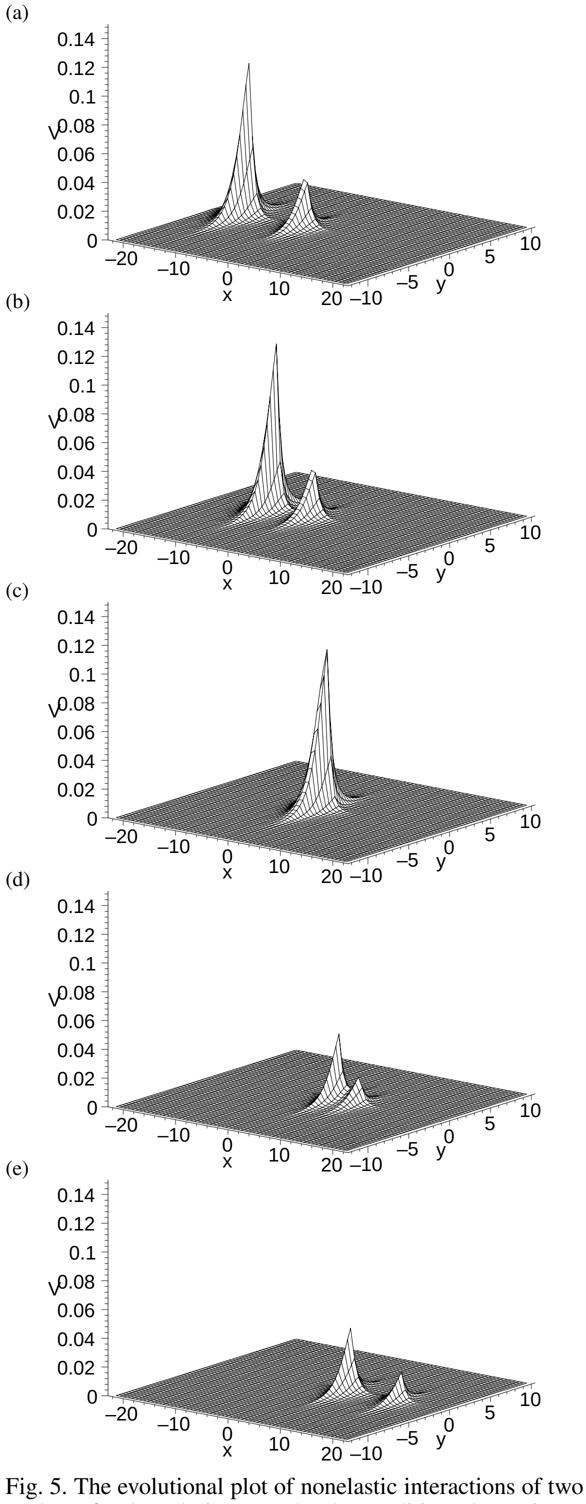

Fig. 5. The evolutional plot of nonelastic interactions of two peakons for the solution V under the condition (51) at times: (a) $t = -15$, (b) $t = -10$, (c) $t = 0$, (d) $t = 6$, (e) $t = 14$.

two peakons move in the same direction, but their velocities are different. One peakon catches up with the other and they will be in collision with each other. From Fig. 5 we can see that the shapes and amplitudes of two peakons are changed after collision. What's more, after their departure, the distance of the two peakons becomes larger.

5. Summary and Discussion

Via conditional similarity reduction, we successfully transformed the (2+1)-dimensional DLW system to a special situation, i. e. the Riccati equation $(\phi' = a_0 + a_1\phi + a_2\phi^2)$, and some new exact solutions to the DLW system were derived. Here the special situation means that in order to find exact excitations of the NEE, we transformed the NEE to some special reduction equations we wanted in advance. For instance, we tried to transform the (2+1)-dimensional DLW system to the Riccati equation, then separated R_3 and R_5 into r_{31}, r_{32} and r_{51}, r_{52}, respectively, and separated (5) into three parts, such as (16). In the fourth part of the paper, based on the solution v_3 (40), we obtained some special dromion and peakon excitations and discussed the interactions between two solitons. Especially, the phenomena showed in Fig. 5 of two peakons running after each other and in collision with each other have never been reported before.

Acknowledgements

The authors would like to thank Professor Jie-Fang Zhang for his fruitful and helpful suggestions. This work has been supported by the Natural Science Foundation of Zhejiang Province (Grant No. Y606128), the Natural Science Foundation of Zhejiang Lishui University (Grant No. KZ08001), and the Scientific Research Fund of Zhejiang Provincial Education Department of China (Grant No. 20070568).

[1] S. Lie, J. Arch. Math. **6**, 328 (1881).
[2] G. W. Bluman and J. D. Cole, J. Math. Mech. **18**, 1025 (1969).
[3] P. A. Clarkson and M. D. Kruskal, J. Math. Phys. **30**, 2201 (1989).
[4] P. A. Clarkson and S. Hood, Eur. J. Appl. Math. **3**, 381 (1992).
[5] S. Y. Lou, Phys. Lett. A **151**, 133 (1990).
[6] X. Y. Tang, J. Lin, and S. Y. Lou, Commun. Theor. Phys. **35**, 399 (2001).
[7] S. Y. Lou and X. Y. Tang, Chin. Phys. **10**, 897 (2001).
[8] S. Y. Lou, X. Y. Tang, and J. Lin, J. Math. Phys. **41**, 8286 (2000).
[9] S. Wang, X. Y. Tang, and S. Y. Lou, Chaos, Solitons, and Fractals **19**, 769 (2004).
[10] S. Wang, Y. S. Xu, and S. Y. Lou, Chin. Phys. **14**, 1049 (2003).
[11] J. P. Fang and C. L. Zheng, Z. Naturforsch. **60a**, 245 (2005).
[12] C. L. Zheng and Z. M. Sheng, Int. J. Mod. Phys. B **17**, 4407 (2003).
[13] J. P. Fang and C. L. Zheng, Acta Phys. Sin. **54**, 2991 (2005) (in Chinese).
[14] C. L. Chen and Y. S. Li, Theor. Phys. **38**, 129 (2002).
[15] J. P. Fang and C. L. Zheng, Acta Phys. Sin. **54**, 670 (2005) (in Chinese).
[16] J. P. Fang and C. L. Zheng, Commun. Theor. Phys. **44**, 203 (2005).
[17] S. H. Ma and J. P. Fang, Acta Phys. Sin. **55**, 5611 (2006) (in Chinese).
[18] S. H. Ma, X. H. Wu, J. P. Fang, and C. L. Zheng, Z. Naturforsch. **61a**, 249 (2006).
[19] S. H. Ma, J. P. Fang, and C. L. Zheng, Z. Naturforsch. **62a**, 8 (2007).
[20] S. H. Ma, J. Y. Qiang, and J. P. Fang, Acta Phys. Sin. **56**, 0620 (2007) (in Chinese).
[21] S. H. Ma, J. P. Fang, and H. P. Zhu, Acta Phys. Sin. **56**, 4319 (2007) (in Chinese).
[22] M. Boiti, J. J. P. Leon and F. Pempinelli, Inverse Probl. **3**, 371 (1987).
[23] G. Paquin and P. Winternitz, Physica D **46**, 122 (1990).
[24] S. Y. Lou, J. Phys. A **27**, 3235 (1994).
[25] S. Y. Lou, Math. Meth. Appl. Sci. **18**, 789 (1995).